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# Nondestructive Determination of Buckling for Plates and Bars Including Shear

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**It will be shown that the generic integral equation for nondestructive determination of the buckling loads obtained previously (Baruch, M., "Integral Equations for Nondestructive Determination of Buckling Loads for Elastic Bars and Plates," *Israel Journal of Technology*, Vol. 11, No. 1–2, 1973, pp. 1–8) includes the influence of the transverse shear strains on the buckling load. This can not be said for the integral equations obtained by differentiation of the generic integral equation.**

## Introduction

THE integral equations for nondestructive determination of buckling loads for elastic plates and bars have been derived, for example, in Ref. 1 (in which one can find many references connected with the discussed subject). These equations have been used for nondestructive determination of elastic columns,[1,2] elastic flat plates,[3] and damaged composite plates.[4] Piche[5] presented an interesting theoretical development using integral equations. Segall and Baruch[2] applied an integral equation [Eq. (7) of Ref. 1] that was obtained by differentiation of the generic integral equation [Eq. (6) of Ref. 1]. The applied integral equation in Ref. 2 has symmetric positive definite Green function and hence, nice analytical and calculation properties.[1] However, as will be shown later, the symmetric Green function, obtained by differentiation of the nonsymmetrical generic Green function, does not include the influence of the shear strains, whereas the generic Green function does. Fortunately, the tested column in Ref. 2 was quiet flexible so that one could ignore the influence of the shear strains on the magnitude on the buckling load. On the other hand, Segall and Springer[3] and Salunkhe and Mujumdar[4] applied intuitively the generic integral equation, and in this way they included automatically the influence of the shear stains on the magnitude of the buckling load. Note that the shear strains can be important, especially for structures of composite materials, even more so when the structure is damaged. Hence, it is important to show theoretically that the generic integral equation takes into account the shear strains. This is the main purpose of this paper.

## Generic Integral Equation for Buckling Load of Columns

In Ref. 1 the generic integral equation for buckling load of columns was obtained by applying the Bernoulli–Euler law (see Ref. 6) for the deformation of the beam section. In this way, the influence of the shear strains on the deformation of the beam is ignored. Here, the generic integral equation will be obtained by applying the Timoshenko[7,8] assumption on the effect of the shearing force on the critical load of the column (also see Refs. 9–12).

A Timoshenko-type column is constrained on the boundaries by coupled elastic springs. The springs are defined by the spring constant matrices $\alpha$ and $\beta$ for $s = 0$ and 1, respectively,

$$\alpha = \begin{bmatrix} \alpha_{11}, \alpha_{12} \\ \alpha_{21}, \alpha_{22} \end{bmatrix} \tag{1}$$

$$\beta = \begin{bmatrix} \beta_{11}, \beta_{12} \\ \beta_{21}, \beta_{22} \end{bmatrix} \tag{2}$$

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where $\alpha$ and $\beta$ are symmetric positive matrices[13] (Fig. 1). Axial forces load the column. The stability equations and the boundary conditions will be obtained by using the principle of minimum additional potential energy during buckling.[14] In view of these assumptions one obtains[1]

$$\delta(U + V) = 0 \tag{3}$$

where $U$ is the strain energy and $V$ is the potential of the external load:

$$V = -\frac{1}{2}\int_0^l \left[ P_1 + \int_0^s p(\eta)\,\mathrm{d}\eta \right] \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 \mathrm{d}s$$

$$= -\frac{1}{2}\int_0^l N(s) \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 \mathrm{d}s \tag{4}$$

where $l$ is the span of the beam and $N(s)$ is the varying axial force assuming compression to be positive.[1] Hence,

$$\delta(U + V) = \delta U - \int_0^l \left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right) \delta\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right) \mathrm{d}s = 0 \tag{5}$$

The strain energy for a Timoshenko beam is given by

$$U = \frac{1}{2}\int_0^l EI\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2 \mathrm{d}s + \frac{1}{2}\int_0^l kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s} - \psi\right)^2 \mathrm{d}s$$

$$+ \frac{1}{2} J_0^t \alpha J_0 + \frac{1}{2} J_l^t \beta J_1 \tag{6}$$

$E$ is the varying modulus of elasticity, $I$ is the varying moment of inertia, $\psi$ is the rotation of the cross section of the beam, $k$ is the varying shear factor, $G$ is the varying shear modulus, $A$ is the varying cross section of the beam, and $y$ is its deflection. For an interesting paper, connected with Timoshenko's shear coefficient, see Hutchinson.[15] Note that here the Green function $v(x, s)$ is obtained from experiments, and the evaluation of the shear factor $k$ is not needed. Here $(\,)^t$ represents the transpose of a vector and $J$ is given as follows:

$$J = \begin{Bmatrix} \psi \\ y \end{Bmatrix} \tag{7}$$

By performing the variations, one obtains the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s} - \psi\right)\right] - \frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(EI\frac{\mathrm{d}\psi}{\mathrm{d}s}\right) + kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s} - \psi\right) = 0 \tag{8}$$

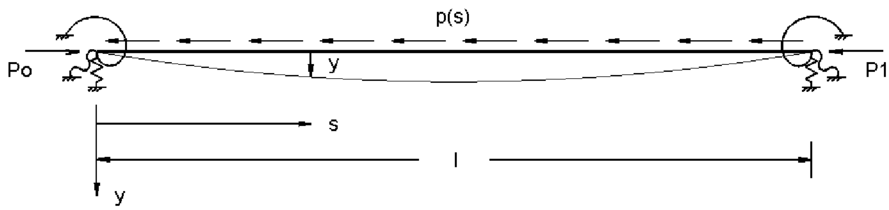


**Fig. 1 Elastically constrained Timoshenko beam loaded by axial forces.**

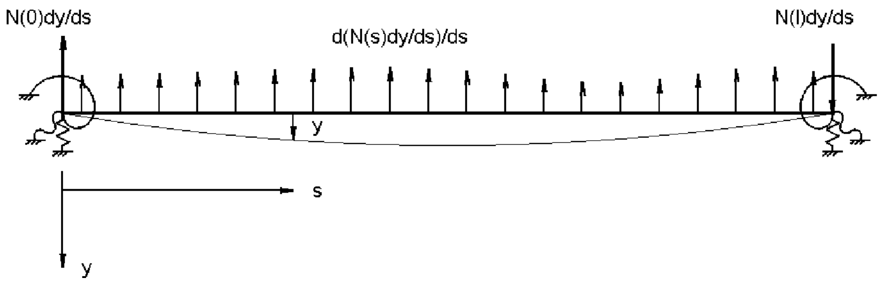


**Fig. 2 Equivalent Timoshenko beam.**

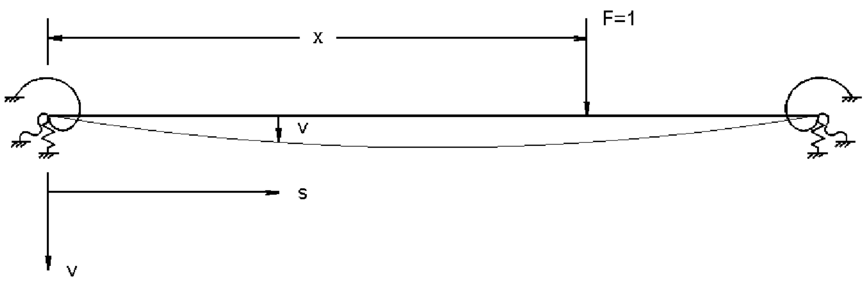


**Fig. 3 Beam loaded by a unit force.**

with the boundary conditions, for $s = 0$,

$$-EI\frac{\mathrm{d}\psi}{\mathrm{d}s} + \alpha_{11}\psi + \alpha_{12}y = 0 \qquad \text{or} \qquad \psi = 0$$

$$-kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s} - \psi\right) + \alpha_{21}\psi + \alpha_{22}y + N\frac{\mathrm{d}y}{\mathrm{d}s} = 0 \qquad \text{or} \qquad y = 0 \tag{9}$$

and, for $s = l$,

$$EI\frac{\mathrm{d}\psi}{\mathrm{d}s} + \beta_{11}\psi + \beta_{12}y = 0 \qquad \text{or} \qquad \psi = 0$$

$$kGA\left(\frac{\mathrm{d}y}{\mathrm{d}s} - \psi\right) + \beta_{21}\psi + \beta_{22}y - N\frac{\mathrm{d}y}{\mathrm{d}s} = 0 \qquad \text{or} \qquad y = 0 \tag{10}$$

Note that the eigenvalue appears, as expected, in the boundary conditions, as well as in the stability equations.

Now, the Timoshenko column loaded by axial forces, which generally appear in the boundary as well, will be replaced by an equivalent Timoshenko beam loaded by a fictitious external load of distributed forces of magnitude $-\mathrm{d}(N\mathrm{d}y/\mathrm{d}s)/\mathrm{d}s$ per unit length and two fictitious concentrated forces of magnitude, $-N(0)\mathrm{d}y/\mathrm{d}s$ and $+N(l)\mathrm{d}y/\mathrm{d}s$, respectively. Note that the concentrated forces are external loads and not boundary conditions. The spring constants on the boundaries have the same values as in the case of the original column (Fig. 1). To avoid the necessity to distinguish between the rotation of the cross section of the beam and the first derivative of its deflection, the equivalent beam proposed here is different from the one proposed in Ref. 1. Note that, in the case of the equivalent beam, no axial forces appear in the boundary conditions (Fig. 2).

It can be easily shown that the Timoshenko column of Fig. 1 and the Timoshenko beam of Fig. 2 are equivalent. To find the equilibrium equations of the equivalent Timoshenko beam, one can again use the principle of minimum potential energy:

$$\delta(U+V) = \delta U - \left[N\frac{\mathrm{d}y}{\mathrm{d}s}\delta y\bigg|_0^l - \int_0^l \frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right)\delta y\,\mathrm{d}s\right] \tag{11}$$

Note that the strain energy is exactly the same as in the case of the Timoshenko column. By performing the integration, one obtains

$$\delta(U+V) = \delta U - \int_0^l \left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right)\delta\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)\mathrm{d}s = 0 \tag{12}$$

which is exactly the same expression obtained for the Timoshenko column [see Eq. (5)]. It is clear that, after performing the necessary manipulations, one would obtain for the equivalent Timoshenko beam the same differential equations (9) and the same boundary conditions (10) and (11) as for the original Timoshenko column.

In the case of the equivalent Timoshenko beam (Fig. 2) there does not appear any axial force, and the beam can be treated formally as a linear structure. In this case the Maxwell–Betti reciprocal theorem (see Sokolnikoff[6] and Hoff[14]) is applicable and will be used to find the kernel of the integral equation for the Timoshenko column.

The elastically restrained Timoshenko beam is loaded once by the distributed load $-\mathrm{d}(N\mathrm{d}y/\mathrm{d}s)/\mathrm{d}s$ and the two concentrated loads $-N(0)\mathrm{d}y/\mathrm{d}s$ and $+N(l)\mathrm{d}y/\mathrm{d}s$ (Fig. 2) and once by a unit force $F = 1$ (Fig. 3).

By applying the Maxwell–Betti reciprocal theorem, one obtains

$$y(x) = N\frac{\mathrm{d}y}{\mathrm{d}s}v\bigg|_0^l - \int_0^l \frac{\mathrm{d}}{\mathrm{d}s}\left[N(s)\frac{\mathrm{d}y(s)}{\mathrm{d}s}\right]v(x,s)\,\mathrm{d}s \tag{13a}$$

or from Eq. (13a),

$$y(x) = \int_0^l \frac{\partial v(x,s)}{\partial s} N(s) \frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s \tag{13b}$$

where $y(x)$ is the deflection of the buckled column at point $x$ due to the compressive forces (Fig. 1) and $v(x, s)$ is the deflection of the beam at point $s$ due to a unit force at point $x$.

By representation of $N(s)$ as

$$N(s) = \lambda n(s) \tag{14}$$

where $\lambda$ is the required eigenvalue and $n(s)$ is a known function, one obtains

$$y(s) = \lambda \int_0^l \frac{\partial v(x,s)}{\partial s} n(s) \frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s \tag{15}$$

Equation (13b) or its equivalent equation (15) is the required integral equation for the calculation of the buckling load and is exactly the same one obtained by applying the Bernoulli–Euler law for beams [see Ref. 1, Eq. (6)]. Note that Eq. (13b) was obtained here for the Timoshenko beam, and hence, it takes into account the influence of the shear strain on the magnitude of the buckling load. The kernel $v(x, s)$ can be found experimentally, for example, by applying a force $F$ at point $x$ and dividing the measured displacements at point $s$ by $F$. Note that $v(x, s)$ satisfies the boundary conditions, and therefore, their determination is superfluous. Once the kernel is known, one can use one of the powerful numerical techniques of the integral equations to calculate the buckling load.[2–4,16–20]

Differentiation of Eq. (13b) in respect to $x$ will give [see Ref. 1, Eq. (7)]

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \int_0^l \frac{\partial^2 v(x,s)}{\partial x \partial s} N(s) \frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s \tag{16}$$

As shown in Ref. 1, when used by applying the Bernoulli–Euler law for beams, Eq. (16) has very nice mathematical properties. However, when the kernel $\partial v(x, s)/\partial s$ includes the transverse shear force, it becomes discontinuous in respect to $s$ and especially in respect to $x$ (the place of the unit force). As will be shown, by differentiation of $v(x, s)$ in respect to $s$ and $x$, the influence of the shear force on the magnitude of the buckling load can be partially or even entirely lost. It will be also shown that, by differentiation of $v(x, s)$ in respect to $s$ only, the influence of the shear force on the magnitude of the buckling load is still fully preserved. Hence, as already indicated, Eq. (15) is a fundamental expression and must be used as it is.

**Step 1: Cantilever Beam Column**

A constant axial force $N$ loads a cantilever beam column with constant $EI$ and constant $kAG$ (Fig. 4). In this case $n(s) = 1$, and Eq. (15) becomes

$$y(x) = N \int_0^l \frac{\partial v(x,s)}{\partial s} \frac{\mathrm{d}y(s)}{\mathrm{d}s}\,\mathrm{d}s \tag{17}$$

where $N = N_{cr}$ is the critical load that causes the cantilever to buckle.

To calculate the kernel (Fig. 5), we will apply the basic constitutive expressions of the Timoshenko beam theory

$$v = Vm + Vs \tag{18}$$

$$EI\frac{\mathrm{d}^2 Vm}{\mathrm{d}s^2} = EIVm,ss = -M, \qquad kGA\frac{\mathrm{d}Vs}{\mathrm{d}s} = kGAVs,s = S \tag{19}$$

where $Vm$ and $Vs$ are the displacements caused by the moment $M$ and the shear force $S$, respectively.

For $s < x$,

$$M = -x + s, \qquad S = 1 \tag{20}$$

The boundary conditions are

$$s = 0 \to \qquad \frac{\mathrm{d}Vm}{\mathrm{d}s} = 0, \qquad v = Vm + Vs = 0 \tag{21}$$

Solution of Eqs. (19) and (20) and fulfillment of the boundary conditions (21) yields

$$v = (1/EI)(xs^2/2 - s^3/6) + s/kGA, \qquad s < x \tag{22}$$

For $s > x$,

$$M = 0, \qquad S = 0 \tag{23}$$

The intermediate conditions are

$$x = s \to \qquad \frac{\mathrm{d}Vm_{\text{left}}}{\mathrm{d}s} = \frac{\mathrm{d}Vm_{\text{right}}}{\mathrm{d}s}, \qquad v_{\text{left}} = v_{\text{right}} \tag{24}$$

By solution of Eqs. (19) and (20) and satisfaction of the intermediate conditions (24), one obtains

$$v = (1/EI)(x^2 s/2 - x^3/6) + x/kGA, \qquad x > s \tag{25}$$

It is clear here that the bending and shear displacements can be separated to obtain, for $s < x$,

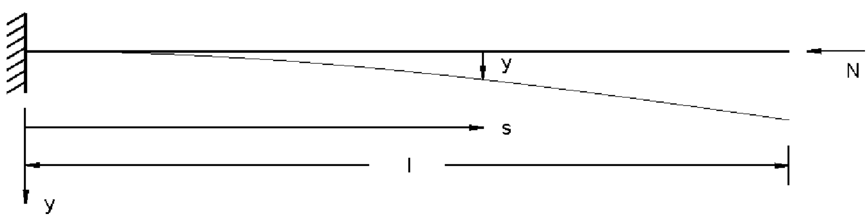


**Fig. 4 Cantilever beam column loaded by a constant axial force.**

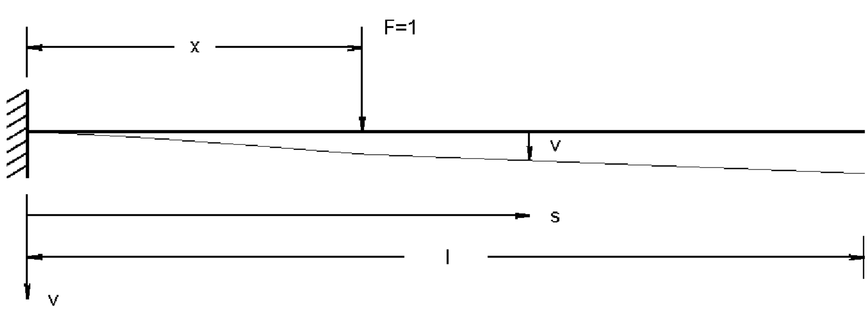


**Fig. 5 Cantilever beam loaded by unit force.**

$$Vm(x,s) = \frac{1}{EI}\left(\frac{xs^2}{2} - \frac{s^3}{6}\right), \qquad Vs(x,s) = \frac{s}{kGA}$$

$$Vm,s = \frac{\partial Vm}{\partial s} = \frac{1}{EI}\left(xs - \frac{s^2}{2}\right), \qquad Vs,s = \frac{\partial Vs}{\partial s} = \frac{1}{kGA}$$

$$Vm,xs = \frac{\partial^2 Vm}{\partial x \partial s} = \frac{s}{EI}, \qquad Vs,xs = \frac{\partial^2 Vs}{\partial x \partial s} = 0 \tag{26}$$

and, for $s > x$,

$$Vm(x,s) = \frac{1}{EI}\left(\frac{x^2 s}{2} - \frac{x^3}{6}\right), \qquad Vs(x,s) = \frac{x}{kGA}$$

$$Vm,s = \frac{\partial Vm}{\partial s} = \frac{x^2}{2EI}, \qquad Vs,s = \frac{\partial Vs}{\partial s} = 0$$

$$Vm,xs = \frac{\partial^2 Vm}{\partial x \partial s} = \frac{x}{EI}, \qquad Vs,xs = 0 \tag{27}$$

One can see from Figs. 6 and 7 that, although $\partial Vm(x,s)/\partial s$ and $\partial^2 Vm(x,s)/\partial x \partial s$ are continuous, $\partial Vs(x,s)/\partial s$ is not and that $\partial^2 Vs/\partial x \partial s$ is here even equivalent to zero. This means that, for the cantilever beam column (Fig. 4), the solution of Eq. (17) will incorporate the influence of the shear strain on the magnitude of the critical force, whereas in the solution of Eq. (16), the shear strain will entirely disappear. Let us check this statement.

One can find that, for the case of a cantilever beam-column with constant $EI$ and $kGA$, loaded by a constant compressive force $N$, the eigenfunction is given[7–12] by

$$y(x) = 1 - \cos(\pi x/2l) \tag{28}$$

It is easy to check that the eigenfunction (28) satisfies both Eqs. (16) and (17). However, the critical loads are different. By utilization of Eq. (16) one gets the classical buckling load obtained by applying the Bernoulli–Euler law (see Refs. 6 and 8):

$$N_{\mathrm{cl1}} = EI(\pi/2l)^2 \tag{29}$$

It is clear that in Eq. (29) the influence of the shear strain entirely disappeared. On the other hand, Eq. (17) yields the buckling load obtained by applying the Timoshenko beam-column theory (see Refs. 7–12),

$$N_{\mathrm{cr1}} = \frac{EI(\pi/2l)^2}{1 + (EI/kGA)(\pi/2l)^2} = \frac{N_{\mathrm{cl1}}}{1 + (N_{\mathrm{cl1}}/kGA)} \tag{30}$$

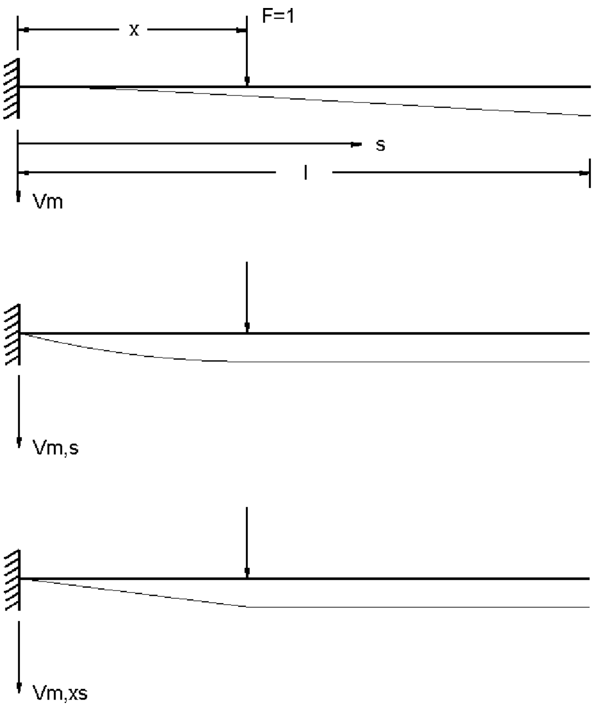


**Fig. 6 Cantilever Timoshenko beam; *Vm* and its derivatives.**

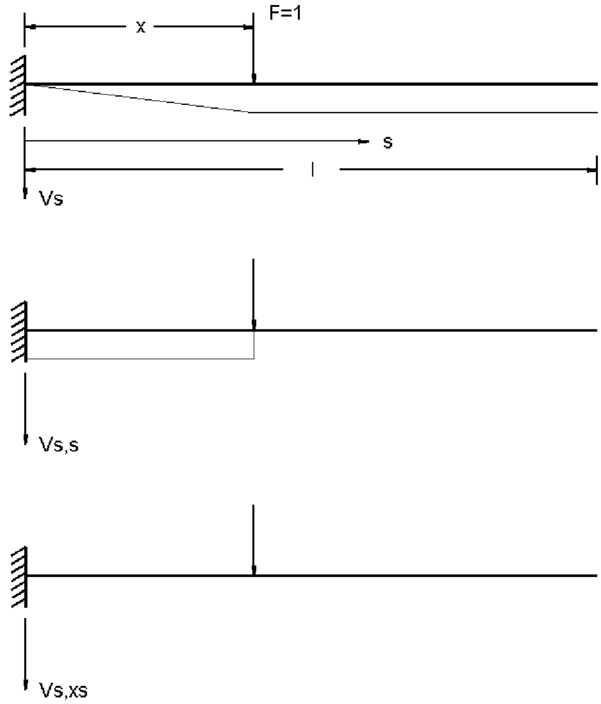


**Fig. 7 Cantilever Timoshenko beam; *Vs* and its derivatives.**

**Step 2: Simply Supported Beam Column**

A constant axial force $N$ loads a simply supported beam-column with constant $EI$ and constant $kAG$ (Fig. 8). In this case, as in the case of the first step, the relevant expression is given in Eq. (17).

To calculate the kernel of a simply supported Timoshenko beam, we will again use Eqs. (18) and (19) (Fig. 9).

For $s < x$,

$$M = s - (xs/l), \qquad S = 1 - (x/l) \tag{31}$$

For $s > x$,

$$M = x - (xs/l), \qquad S = -(x/l) \tag{32}$$

The boundary and intermediate conditions are

$$s = 0 \to \qquad M = 0, \qquad v = 0 \tag{33}$$

$$s = x \to \qquad \frac{\mathrm{d}Vm_{\mathrm{left}}}{\mathrm{d}s} = \frac{\mathrm{d}Vm_{\mathrm{right}}}{\mathrm{d}s}, \qquad v_{\mathrm{left}} = v_{\mathrm{right}} \tag{34}$$

$$s = l \to \qquad M = 0, \qquad v = 0 \tag{35}$$

By solution of Eqs. (19) and (20) and satisfaction of the boundary and intermediate conditions (33–35), one obtains

$$s < x \to \qquad v = \frac{1}{EI}\left(-\frac{s^3}{6} + \frac{xs^3}{6l} - \frac{x^2 s}{2} + \frac{xsl}{3} + \frac{x^3 s}{6l}\right) + \frac{1}{kGA}\left(s - \frac{xs}{l}\right) \tag{36}$$

$$s > x \to \qquad v = \frac{1}{EI}\left(-\frac{x^3}{6} + \frac{x^3 s}{6l} - \frac{xs^2}{2} + \frac{xsl}{3} + \frac{xs^3}{6l}\right) + \frac{1}{kGA}\left(x - \frac{xs}{l}\right) \tag{37}$$

On can see that the bending and the shear displacements can be separated to obtain, for $s < x$,

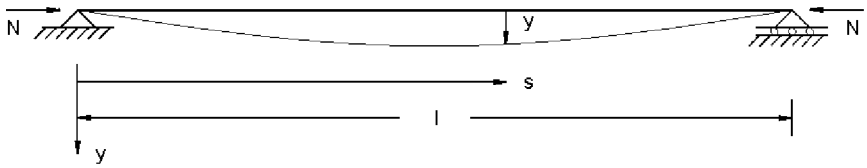


**Fig. 8 Simply supported beam column loaded by a constant force.**

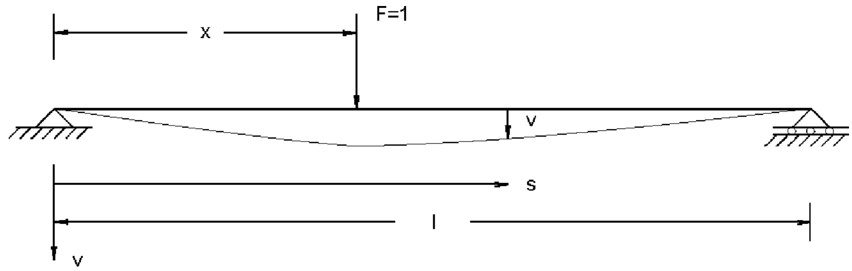


**Fig. 9 Simply supported beam loaded by unit force.**

$$Vm(x,s) = \frac{1}{EI}\left(-\frac{s^3}{6} + \frac{xs^3}{6l} - \frac{x^2 s}{2} + \frac{xsl}{3} + \frac{x^3 s}{6l}\right)$$

$$Vs(x,s) = \frac{1}{kGA}\left(s - \frac{xs}{l}\right)$$

$$Vm,s = \frac{\partial Vm}{\partial s} = \frac{1}{EI}\left(-\frac{s^2}{2} + \frac{xs^2}{2l} - \frac{x^2}{2} + \frac{xl}{3} + \frac{x^3}{6l}\right)$$

$$Vs,s = \frac{\partial Vs}{\partial s} = \frac{1}{kGA}\left(1 - \frac{x}{l}\right)$$

$$Vm,xs = \frac{\partial^2 Vm}{\partial x \partial s} = \frac{1}{EI}\left(\frac{s^2}{2l} - x + \frac{l}{3} + \frac{x^2}{2l}\right)$$

$$Vs,xs = \frac{\partial^2 Vs}{\partial x \partial s} = \frac{1}{kGA}\left(-\frac{1}{l}\right) \tag{38}$$

and, for $s > x$,

$$Vm(x,s) = \frac{1}{EI}\left(-\frac{x^3}{6} + \frac{x^3}{6l} - \frac{xs^3}{2} + \frac{xsl}{3} + \frac{xs^3}{6l}\right)$$

$$Vs(x,s) = \frac{1}{kGA}\left(x - \frac{xs}{l}\right)$$

$$Vm,s = \frac{\partial Vm}{\partial s} = \frac{1}{EI}\left(\frac{x^3}{6l} - xs + \frac{xl}{3} + \frac{xs^2}{2l}\right)$$

$$Vs,s = \frac{\partial Vs}{\partial s} = \frac{1}{kGA}\left(-\frac{x}{l}\right)$$

$$Vm,xs = \frac{\partial^2 Vm}{\partial x \partial s} = \frac{1}{EI}\left(\frac{x^2}{2l} - s + \frac{l}{3} + \frac{s^2}{2l}\right)$$

$$Vs,xs = \frac{\partial^2 Vs}{\partial x \partial s} = \frac{1}{kGA}\left(-\frac{1}{l}\right) \tag{39}$$

One can see again (Fig. 10) that, although $\partial Vm(x,s)/\partial s$ and $\partial^2 Vm(x,s)/\partial x \partial s$ are continuous, $\partial Vs(x,s)/\partial s$ is not. Here again one expects that the solution of Eq. (17) would incorporate the influence of the shear strain on the magnitude of the critical force and that in the solution of Eq. (16) the shear strain would disappear despite that (Fig. 11), surprisingly, $\partial^2 Vs(x,s)/\partial x \partial s$ is continuous (constant). Again, let us check this statement.

One can find that, for the case of a simply supported beam-column with constant $EI$ and $kGA$, loaded by a constant compressive force $N$, the eigenfunction is given by

$$y(x) = \sin(\pi x/l) \tag{40}$$

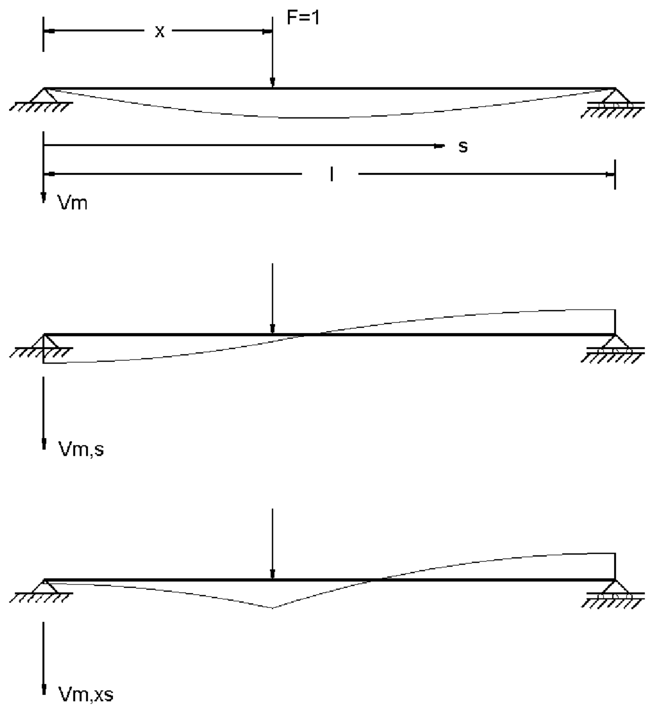


**Fig. 10 Simply supported Timoshenko beam; *Vm* and its derivatives.**

As before, one can find that expression (40) satisfies both Eqs. (16) and (17). However, again, the critical loads are different. Utilization of Eq. (16) yields the classical buckling load obtained by applying the Bernoulli–Euler law (see Refs. 6 and 8):

$$N_{cl2} = EI(\pi/l)^2 \tag{41}$$

One can see that in Eq. (41) the influence of the shear strain disappeared. Nevertheless, once again, Eq. (17) yields the buckling load obtained by applying the Timoshenko beam-column theory[7,8] (also see Refs. 9–12),

$$N_{cr2} = \frac{EI(\pi/l)^2}{1 + (EI/kGA)(\pi/l)^2} = \frac{N_{cl2}}{1 + (N_{cl2}/kGA)} \tag{42}$$

**Step 3: General Case**

From steps 1 and 2, it is clear that, due to the shear strain, $\partial v(x,s)/\partial s$ will also be discontinuous for the general case of a Timoshenko beam. Hence, by use of Eq. (16), the influence of the shear strain on the magnitude of the buckling load may diminish or even disappear entirely. To ensure the full participation of the shear strain on the magnitude of the buckling load, one has to use generic equations (13) and (15).

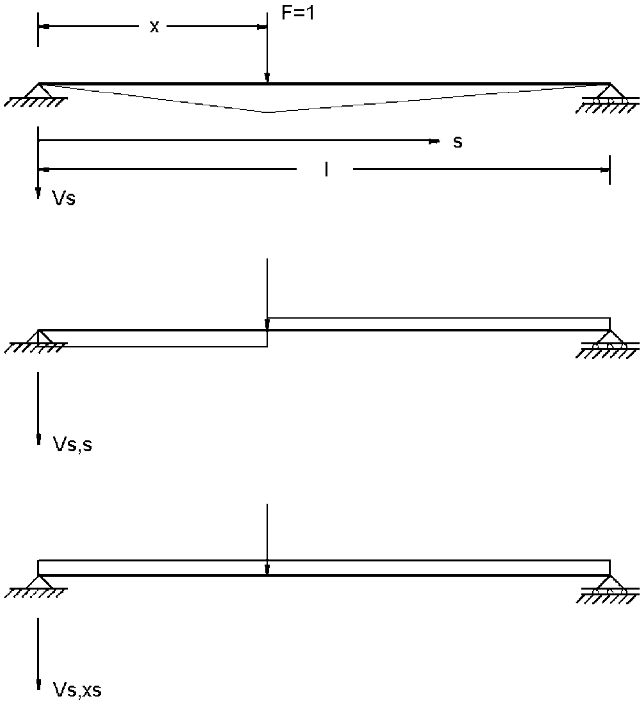


**Fig. 11 Simply supported Timoshenko beam; *Vs* and its derivatives.**

## Formulation of the Integral Equations for Buckling of Plates Using First-Order Shear Theories

Following the approach for the Timoshenko beam given earlier and the approach given in Ref. 1, it is clear that the generic integral equation for the buckling of a Mindlin–Reissner plate will be that obtained by using the classical Kirchhoff plate theory. Note that the Mindlin–Reissner plate theory is an extension of the Timoshenko beam theory. For completeness, we will repeat here the generic integral equation for the buckling of plates given by Eq. (22) of Ref. 1:

$$w(x,s) = \iint_A \times \left\{ \begin{aligned} &\frac{\partial R(x,y;\xi,\eta)}{\partial \xi}\left[N_x(\xi,\eta)\frac{\partial w(\xi,\eta)}{\partial \xi} + N_{xy}(\xi,\eta)\frac{\partial w(\xi,\eta)}{\partial \eta}\right] \\ &+\frac{\partial R(x,y;\xi,\eta)}{\partial \eta}\left[N_y(\xi,\eta)\frac{\partial w(\xi,\eta)}{\partial \eta} + N_{xy}(\xi,\eta)\frac{\partial w(\xi,\eta)}{\partial \xi}\right] \end{aligned} \right\} \times \mathrm{d}A \tag{43}$$

where $w(x, y)$ is the buckling mode shape, $R(x, y; \xi, \eta)$ represents the deflections of the plate caused by a unit force applied at $(x, y)$, $(\xi, \eta)$ are the running Cartesian coordinates, and $N_x$, $N_y$, and $N_{xy}$ are the membrane varying forces that yield buckling. Compression is assumed positive. It must be emphasized that $R(x, y; \xi, \eta)$ automatically fulfills the required boundary conditions.

A special case arises when a constant force acts in one direction. For example, if $N_x = N_{x0}$ is constant and $N_y$ and $N_{xy}$ are zero, one obtains, from Eq. (43),

$$w(x,y) = N_{x0} \iint_A \frac{\partial R(x,y;\xi,\eta)}{\partial \xi}\frac{\partial w(\xi,\eta)}{\partial \xi}\,\mathrm{d}A \tag{44}$$

To include the influence of the transfer shear strain on the magnitude of the buckling load one must use the generic equations (43) and (44) and not their derivatives.

## Discussion

Eisenberger[21] found that the differences in the deflections calculated by using Timoshenko beam theory and high-order beam theory[22–27] "are for most cases insignificant."[21] Khdeir and Reddy[27] came to a similar conclusion for the buckling load of beam-column. Hence, it is natural to make the following assumption:

It is conjectured that the generic integral equations (13) and (43) are valid also for high-order shear theories.

## Conclusions

It was shown that the generic integral equations (13) and (43), but not their derivatives, include the influence of the shear strain on the magnitude of the buckling load.

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